Warsaw University of Technology

Faculty of Power and Aeronautical Engineering

WARSAW UNIVERSITY OF TECHNOLOGY

Institute of Aeronautics and Applied Mechanics

Finite element method (FEM)

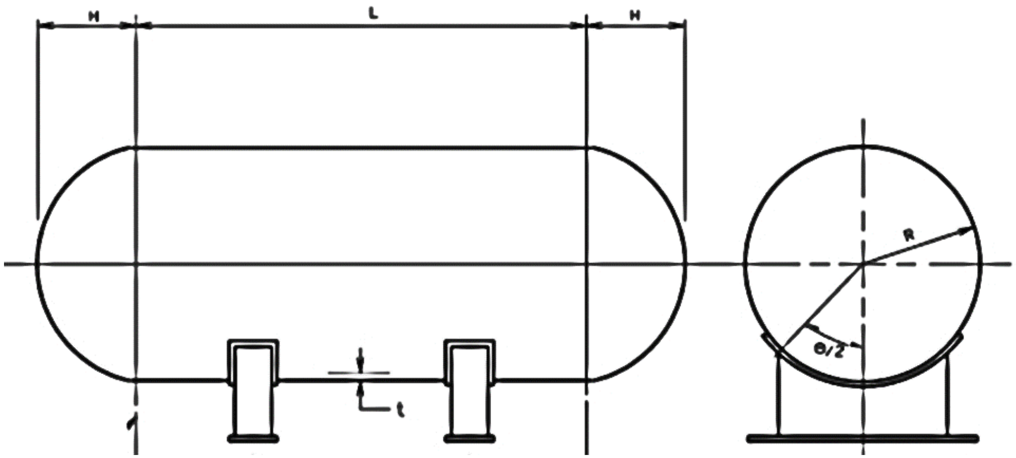
3D shell finite element

06.2021

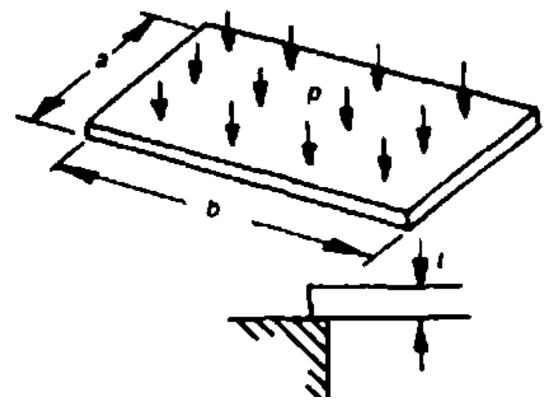
Shells and plates

Thin-shells and plates models can be applied to analyze the following constructions:

- aircraft fusalage, the wing cover,
- boat hull,
- roof (floor) of building.



Thin shell of revolution



Rectangular plate

Examples of plates and shells

aircraft skins (shell model)

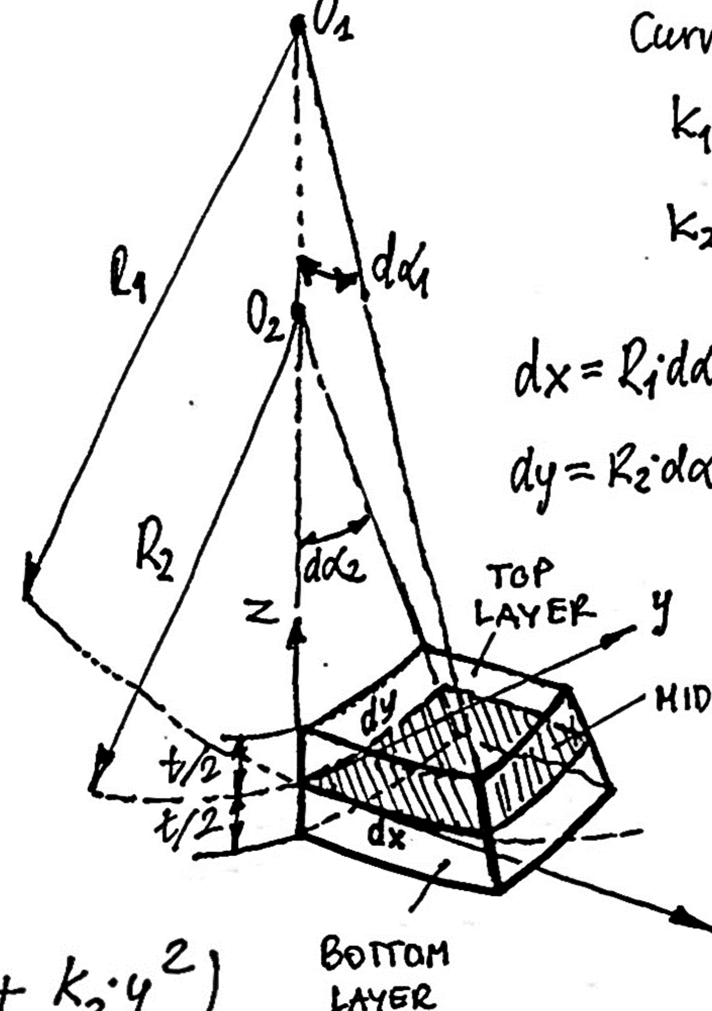
construction of a building (plate model)

a motor yacht:
the hull (shell), the deck (plate)

LINEAR THEORY OF THIN SHELLS

types of shells:

- eliptical
- cylindrical
- spherical
- toroidal
- hyperbolic



Curvatures:

$$k_1 = -\frac{1}{R_1}$$

$$k_2 = -\frac{1}{R_2}$$

$$dx = R_1 \cdot d\alpha_1$$

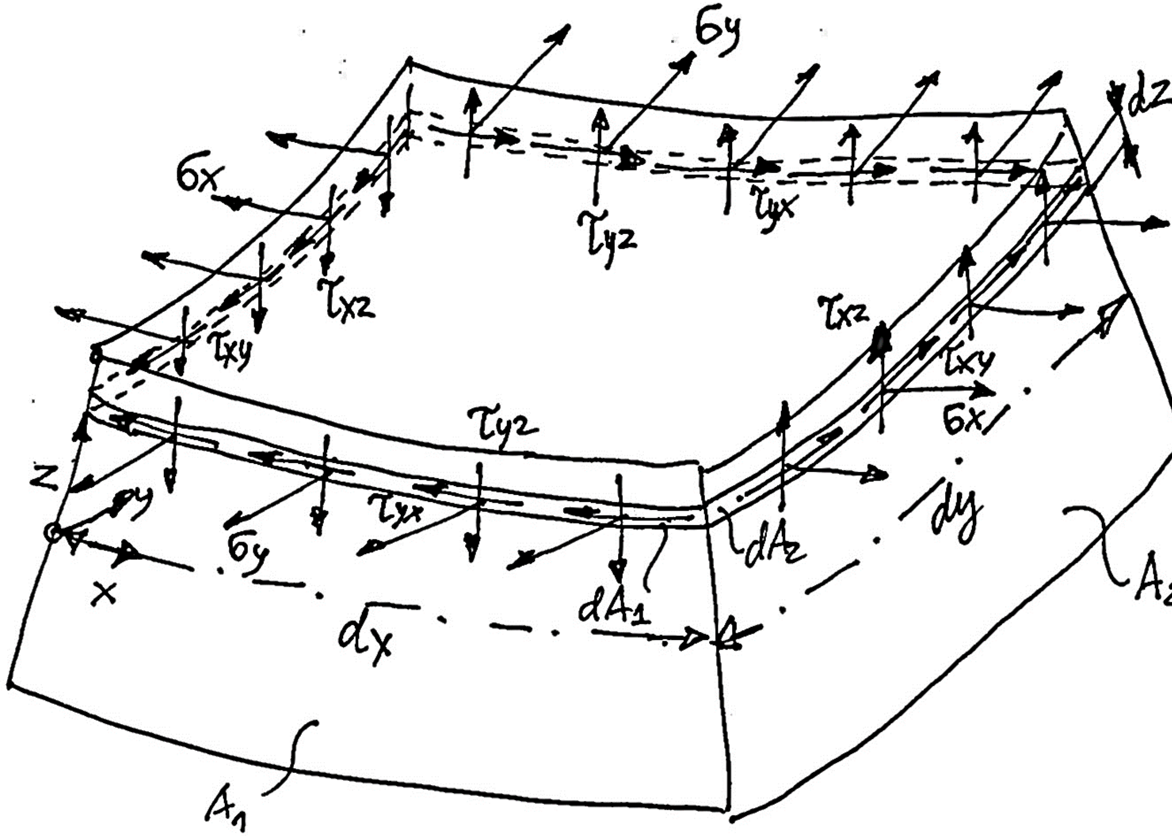
$$dy = R_2 \cdot d\alpha_2$$

$$z = 0.5\left(k_1 \cdot x^2 + 2k_{12} \cdot xy + k_2 \cdot y^2\right)$$

Internal force at level z in a small area dA_2:

$$\sigma_x \cdot dz \cdot (R_2 - z)\, d\alpha_2 = \sigma_x \cdot dz \left(1 - \frac{z}{R_2}\right) R_2\, d\alpha_2 =$$

$$= \sigma_x \left(1 - \frac{z}{R_2}\right) dz\, dy$$



Internal force at level z in a small area dA_2 per unit lenght:

$$dn_x = \frac{\sigma_x\left(1-\frac{z}{R_2}\right)dz\,dy}{dy} = \sigma_x\left(1-\frac{z}{R_2}\right)dz$$

Internal force on a crossectional area A_2 per unit lenght:

$$n_x = \int_{-\frac{t}{2}}^{\frac{t}{2}} \sigma_x\left(1-\frac{z}{R_2}\right)dz \qquad \left(\frac{N}{m}\right)$$

$$\frac{z}{R_1} \approx 0 \quad , \quad \frac{z}{R_2} = 0 \quad \Rightarrow$$

Internal forces

$$n_x = \int_{-\frac{t}{2}}^{\frac{t}{2}} \sigma_x \, dz, \quad n_y = \int_{-\frac{t}{2}}^{\frac{t}{2}} \sigma_y \, dz, \quad n_{xy} = n_{yx} = \int_{-\frac{t}{2}}^{\frac{t}{2}} \tau_{xy} \, dz$$

$$m_x = \int_{-\frac{t}{2}}^{\frac{t}{2}} \sigma_x \cdot z \, dz, \quad m_y = \int_{-\frac{t}{2}}^{\frac{t}{2}} \sigma_y \cdot z \, dz, \quad m_{xy} = m_{yx} = \int_{-\frac{t}{2}}^{\frac{t}{2}} \tau_{xy} \, z \, dz \quad \left(\frac{Nm}{m}\right)$$

$$t_x = \frac{\partial m_x}{\partial x} + \frac{\partial m_{xy}}{\partial y}, \quad t_y = \frac{\partial m_y}{\partial y} + \frac{\partial m_{xy}}{\partial x}$$

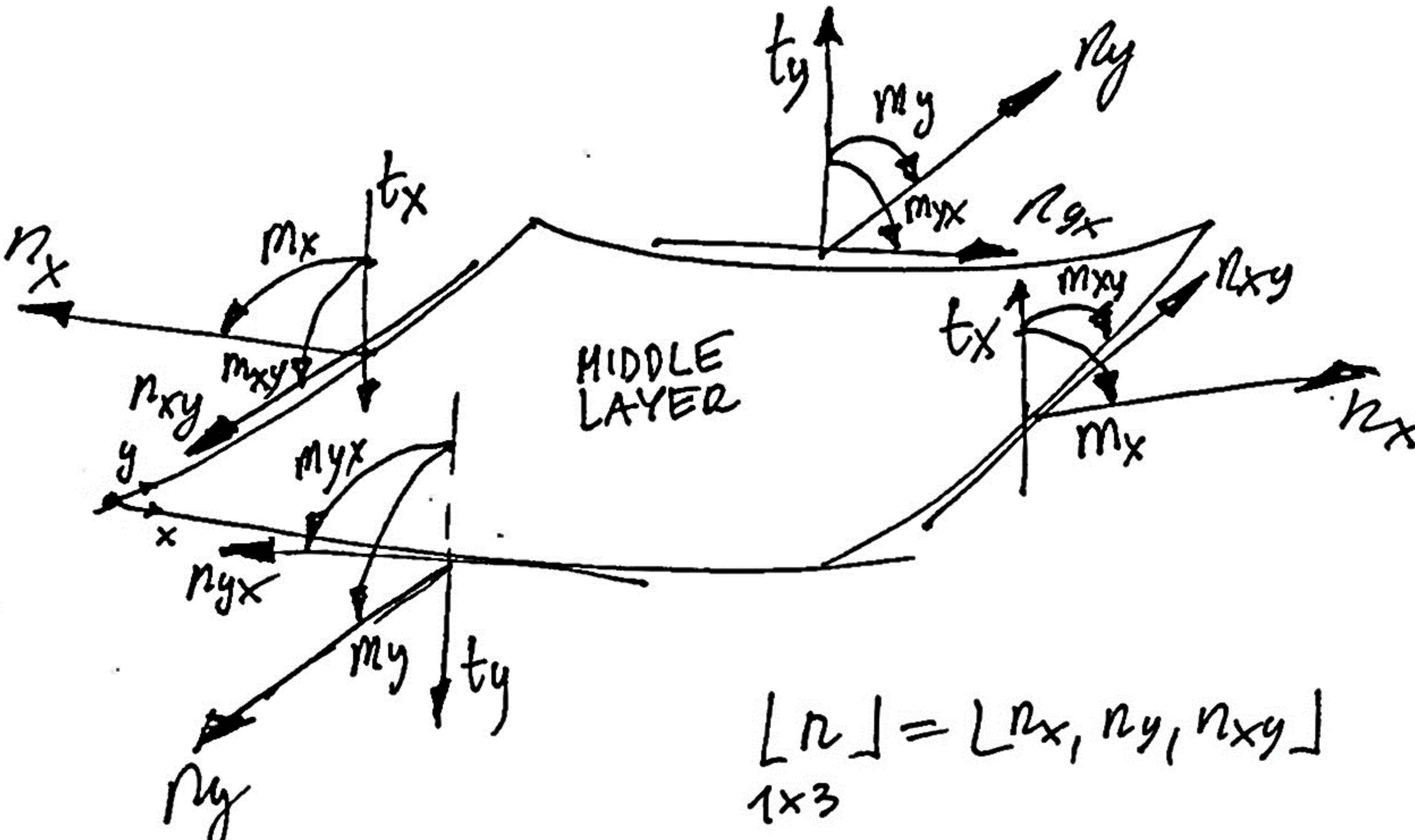
n – normal force per unit lenght

t – shear force per unit lenght

m_x, m_y – bending moments per unit lenght

m_{xy} – torque per unit lenght

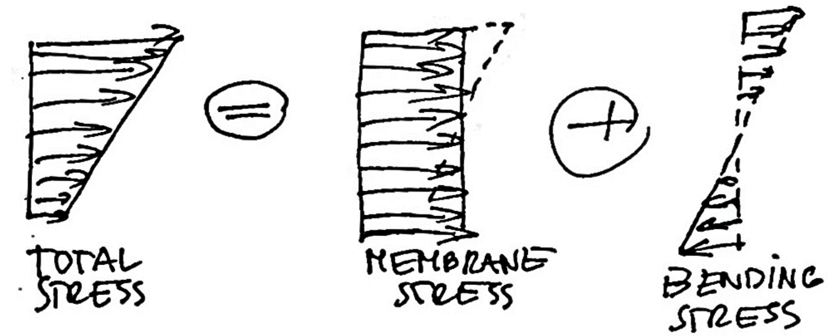
Internal forces

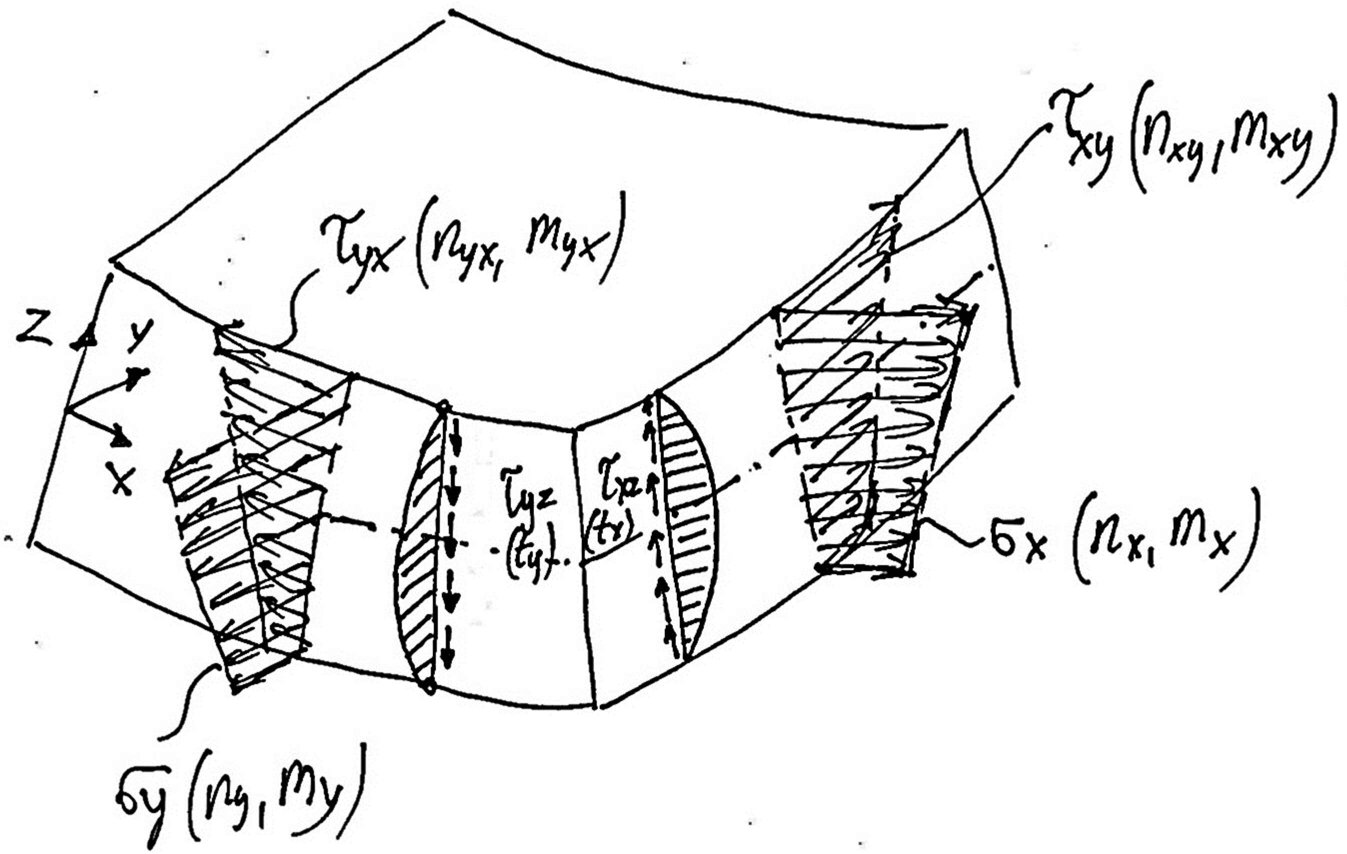


$$\underset{1\times 3}{\lfloor n \rfloor} = \lfloor n_x, n_y, n_{xy} \rfloor$$

$$\underset{1\times 3}{\lfloor m \rfloor} = \lfloor m_x, m_y, m_{xy} \rfloor$$

Stress components

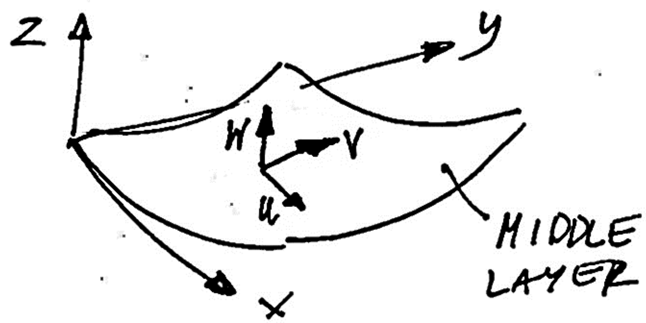




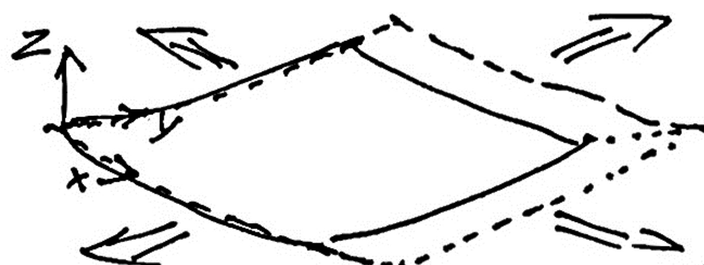
MEMBRANE STRAIN:

1°) deformation of a middle layer in plane xy

2°) deformation of a middle layer along z axis



1°)

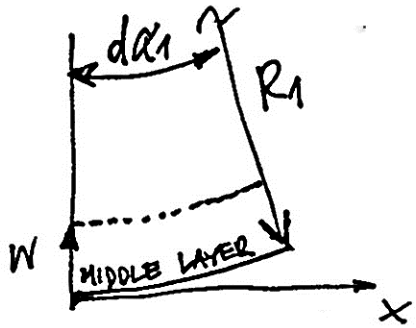


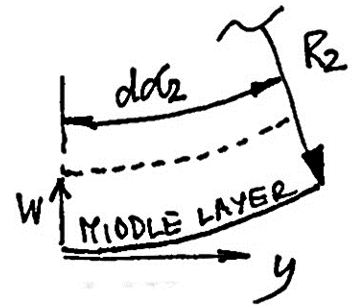
$$\varepsilon_y^{1°} = \frac{\partial v}{\partial y}$$

$$\varepsilon_x^{1°} = \frac{\partial u}{\partial x}$$

MEMBRANE STRAIN

2°)





$$\varepsilon_x^{2°} = \frac{(R_1 - w)\,d\alpha_1 - R_1\,d\alpha_1}{R_1\,d\alpha_1} = -\frac{W}{R_1} = k_1 \cdot W$$

$$\varepsilon_y^{2°} = \frac{(R_2 - w)\,d\alpha_2 - R_2\,d\alpha_2}{R_2\,d\alpha_2} = -\frac{W}{R_2} = k_2 \cdot W$$

MEMBRANE STRAIN

$$1^{\circ}) + 2^{\circ})$$

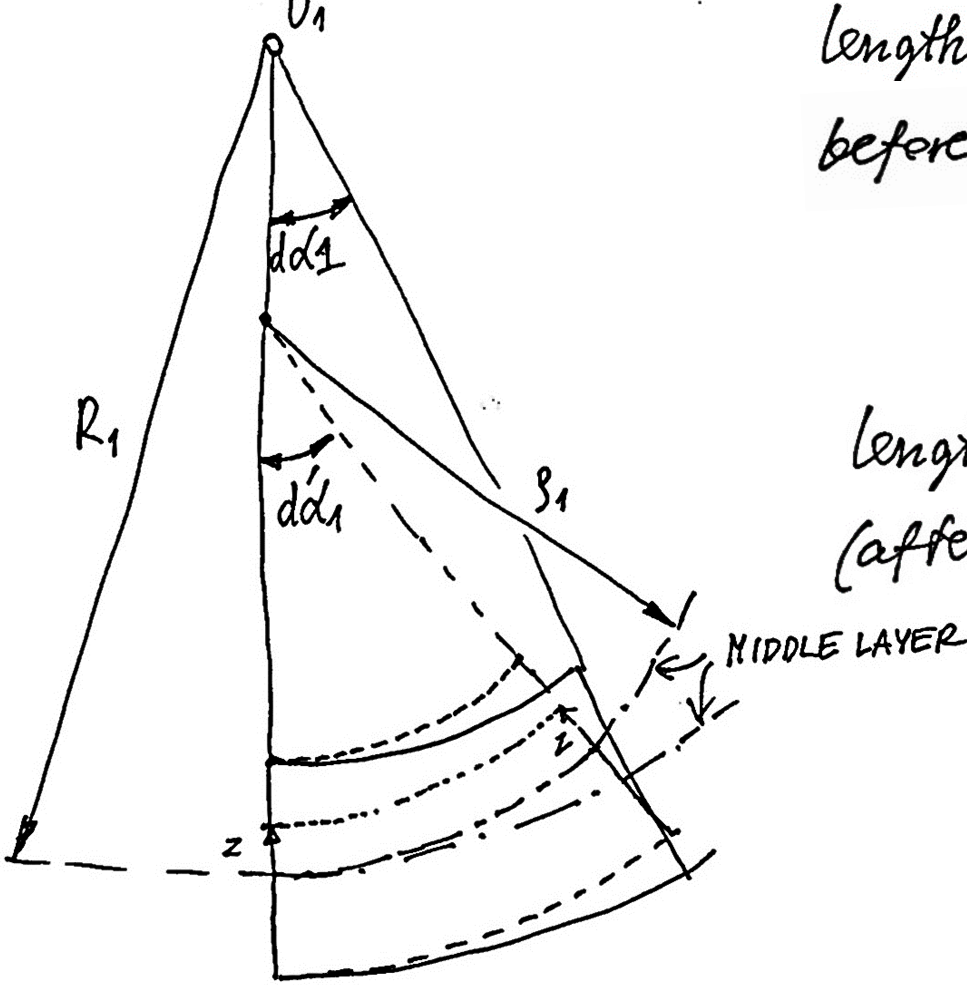
$$\varepsilon_x^{MID} = \frac{\partial u}{\partial x} + k_1 \cdot w$$

$$\varepsilon_y^{MID} = \frac{\partial v}{\partial y} + k_2 \cdot w$$

$$\gamma_{xy}^{MID} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + 2k_{12} \cdot w$$

BENDING STRAIN :

3°) deformation of the layer at level z



length of the layer at level z before deformation:

$$R_1\left(1-\frac{z}{R_1}\right)\cdot d\alpha_1$$

length of the middle layer (after and before deformation):

$$R_1\cdot d\alpha_1 = \rho_1\cdot d\alpha_1'$$

$$\varepsilon_x(z) = \frac{\rho_1\left(1-\frac{z}{\rho_1}\right)d\alpha_1' - R_1\left(1-\frac{z}{R_1}\right)d\alpha_1}{R_1\left(1-\frac{z}{R_1}\right)d\alpha_1} =$$

$$= \frac{\rho_1\left(1-\frac{z}{\rho_1}\right)\frac{R_1}{\rho_1}d\alpha_1 - R_1\left(1-\frac{z}{R_1}\right)d\alpha_1}{R_1\left(1-\frac{z}{R_1}\right)d\alpha_1} =$$

$$= \frac{\left(1-\frac{z}{\rho_1}\right)-\left(1-\frac{z}{R_1}\right)}{\left(1-\frac{z}{R_1}\right)} = \underbrace{\frac{1-\frac{z}{\rho_1}}{1-\frac{z}{R_1}}}_{\approx 1} - 1 = -\frac{z}{\rho_1} =$$

$$= -\frac{\partial^2 w}{\partial x^2}\cdot z = \varkappa_x \cdot z$$

$$\varepsilon_y(z) = -\frac{\partial^2 w}{\partial y^2}\cdot z = \varkappa_y\cdot z \quad ; \quad \gamma_{xy}(z) = -2\frac{\partial^2 w}{\partial x \partial y}\cdot z = \varkappa_{xy}\cdot z$$

TOTAL (MEMBRANE + BENDING) STRAIN VECTOR

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = \begin{Bmatrix} \varepsilon_x^{MID} \\ \varepsilon_y^{MID} \\ \gamma_{xy}^{MID} \end{Bmatrix} + \begin{Bmatrix} \varepsilon_x(z) \\ \varepsilon_y(z) \\ \gamma_{xy}(z) \end{Bmatrix} = \underset{3\times 1}{\left\{ \varepsilon^{MID} \right\}} + \begin{Bmatrix} \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{Bmatrix} \cdot z$$

$$\underset{3\times 1}{\left\{ \varepsilon \right\}} = \underset{3\times 1}{\left\{ \varepsilon^{MID} \right\}} + \underset{3\times 1}{\left\{ \kappa \right\}} \cdot z$$

STRESS COMPONENTS

ASSUMING PLANE STRESS CONDITION

$$\underset{3\times3}{[D]} = \frac{E}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & & \frac{1}{2}(1-\nu) \end{bmatrix}$$

$$\underset{3\times1}{\{\sigma\}} = \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \underset{3\times3}{[D]}\cdot\underset{3\times1}{\{\varepsilon\}} = \underset{3\times3}{[D]}\cdot\underset{3\times1}{\{\varepsilon^{MID}\}} + \underset{3\times3}{[D]}\underset{3\times1}{\{\kappa\}}\cdot z$$

INTERNAL FORCES

$$\underset{3\times1}{\{n\}} = \begin{Bmatrix} n_x \\ n_y \\ n_{xy} \end{Bmatrix} = \int_{-\frac{t}{2}}^{\frac{t}{2}} \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} dz = \underbrace{t \underset{3\times3}{[D]}}_{[D_n]} \cdot \underset{3\times1}{\{\varepsilon^{HID}\}} + \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} = \underset{3\times3}{[D_n]} \cdot \underset{3\times1}{\{\varepsilon^{HID}\}}$$

$$\underset{3\times1}{\{m\}} = \begin{Bmatrix} m_x \\ m_y \\ m_{xy} \end{Bmatrix} = \int_{-\frac{t}{2}}^{\frac{t}{2}} \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} \cdot z\, dz = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} + \frac{t^3}{12} \underset{3\times3}{[D]} \cdot \{\varkappa\} =$$

$$= \frac{Et^3}{12(1-\nu^2)} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1}{2}(1-\nu) \end{bmatrix} \cdot \begin{Bmatrix} \varkappa_x \\ \varkappa_y \\ \varkappa_{xy} \end{Bmatrix} = \underset{3\times3}{[D_m]} \cdot \underset{3\times1}{\{\varkappa\}}$$

STRESS COMPONENTS AS FUNCTIONS OF INTERNAL FORCES

$$\underset{3\times1}{\{\sigma\}} = \underset{3\times3}{[D]}\underset{3\times1}{\{\varepsilon^{MID}\}} + \underset{3\times3}{[D]}\cdot\{\kappa\}\cdot z = \frac{1}{t}\cdot\underset{3\times1}{\{n\}} + \frac{12}{t^3}\{m\}\cdot z$$

$$\{\varepsilon^{MID}\} = \frac{1}{t}\cdot\underset{3\times3}{[D]^{-1}}\cdot\underset{3\times1}{\{n\}} \qquad \{\kappa\} = \frac{12}{t^3}\underset{3\times3}{[D]^{-1}}\cdot\underset{3\times1}{\{m\}}$$

normal stresses:

$$\sigma_x = \frac{n_x}{t} + \frac{12\,m_x}{t^3}\cdot z$$

$$\sigma_y = \frac{n_y}{t} + \frac{12\,m_y}{t^3}\cdot z$$

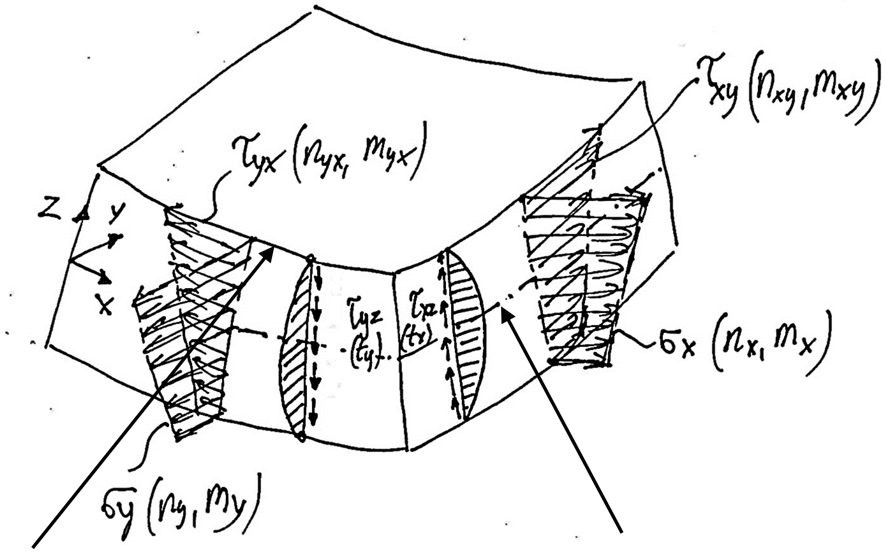
shear stresses:

$$\tau_{xy} = \tau_{yx} = \frac{n_{xy}}{t} + \frac{12\,m_{xy}}{t^3}\cdot z$$

$$\tau_{xz} = \frac{3\,t_x}{2t}\left(1 - \frac{4z^2}{t^2}\right)$$

$$\tau_{yz} = \frac{3\,t_y}{2t}\left(1 - \frac{4z^2}{t^2}\right)$$

MAXIMUM VALUES OF STRESS COMPONENTS



TOP LAYER

$$\sigma_x^{TOP} = \frac{n_x}{t} + \frac{6m_x}{t^2}$$

$$\sigma_y^{TOP} = \frac{n_y}{t} + \frac{6m_y}{t^2}$$

$$\tau_{xy}^{TOP} = \frac{n_{xy}}{t} + \frac{6m_{xy}}{t^2}$$

$$\tau_{xz}^{TOP} = 0$$

$$\tau_{yz}^{TOP} = 0$$

MIDDLE LAYER

$$\sigma_x^{MID} = \frac{n_x}{t}$$

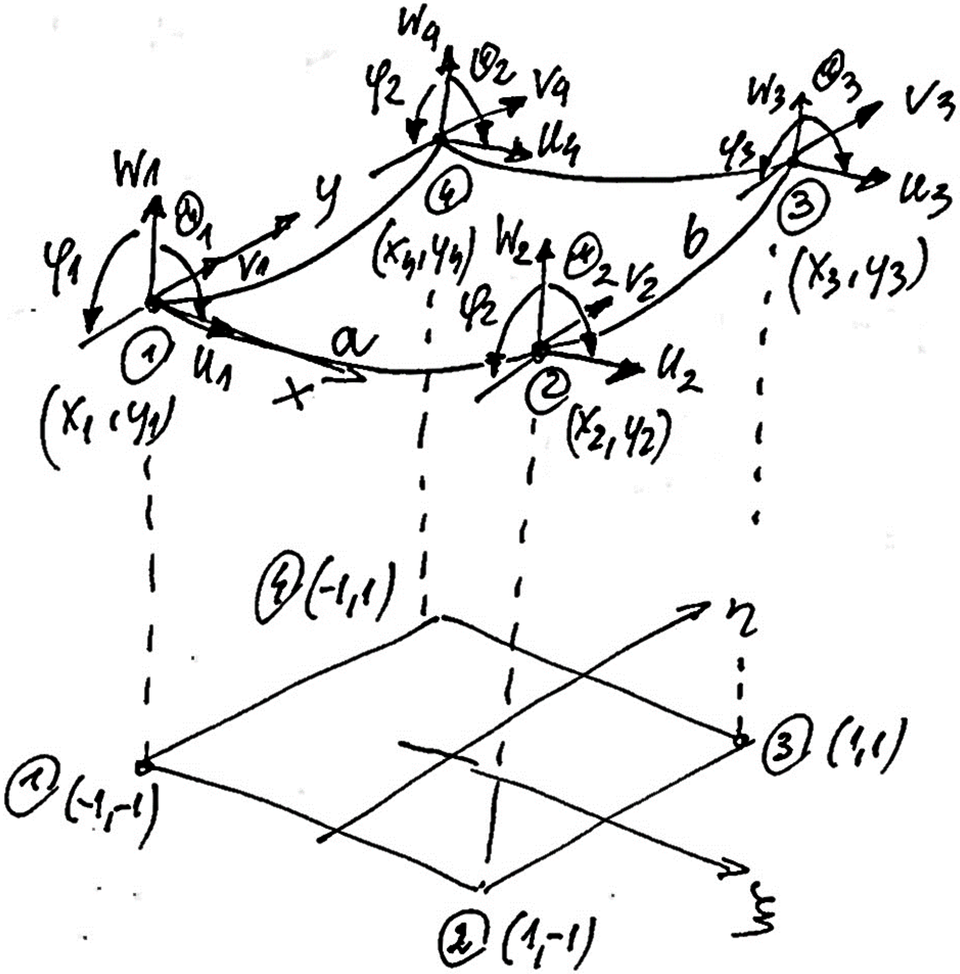
$$\sigma_y^{MID} = \frac{n_y}{t}$$

$$\tau_{xy}^{MID} = \frac{n_{xy}}{t}$$

$$\tau_{xz}^{MID} = \frac{3}{2} \cdot \frac{t_x}{t}$$

$$\tau_{yz}^{MID} = \frac{3}{2} \cdot \frac{t_y}{t}$$

AN ISOPARAMETRIC SHELL FINITE ELEMENT



$$n = 4$$
$$n_0 = 5 \quad \Rightarrow$$
$$n_e = 4 \cdot 5 = 20$$

$$\varphi_i = \left.\frac{dw}{\partial y}\right|_i$$

$$\theta_i = -\left.\frac{\partial w}{\partial x}\right|_i$$

Local vector of nodal parameters (three parts)

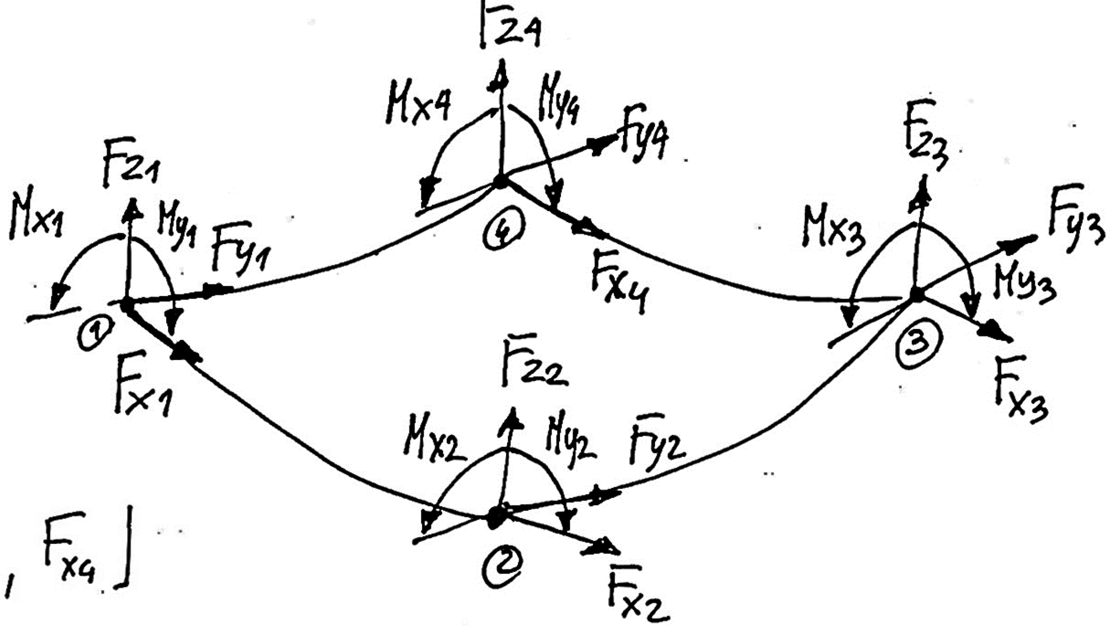
$$\underset{1\times 4}{\lfloor q_u \rfloor_e} = \lfloor u_1, u_2, u_3, u_4 \rfloor_e$$

$$\underset{1\times 4}{\lfloor q_v \rfloor_e} = \lfloor v_1, v_2, v_3, v_4 \rfloor_e$$

$$\underset{1\times 12}{\lfloor q_w \rfloor_e} = \lfloor w_1, \varphi_1, \theta_1, w_2, \varphi_2, \theta_2, w_3, \varphi_3, \theta_3, w_4, \varphi_4, \theta_4 \rfloor_e$$

$$\underset{1\times 20}{\lfloor q \rfloor_e} = \lfloor \lfloor q_u \rfloor_e, \lfloor q_v \rfloor_e, \lfloor q_w \rfloor_e \rfloor_e$$

Local load vector (three parts)



$$\lfloor F_x \rfloor_e = \lfloor F_{x1}, F_{x2}, F_{x3}, F_{x4} \rfloor \quad (1\times 4)$$

$$\lfloor F_y \rfloor_e = \lfloor F_{y1}, F_{y2}, F_{y3}, F_{y4} \rfloor \quad (1\times 4)$$

$$\lfloor F_z \rfloor_e = \lfloor F_{z1}, M_{x1}, M_{y1}, F_{z2}, M_{x2}, M_{y2}, F_{z3}, M_{x3}, M_{y3}, F_{z4}, M_{x4}, M_{y4} \rfloor \quad (1\times 12)$$

$$\lfloor F \rfloor_e = \lfloor \lfloor F_x \rfloor_e, \lfloor F_y \rfloor_e, \lfloor F_z \rfloor_e \rfloor_e \quad (1\times 20)$$

Nodal approximation and shape functions

$$u = N_1 \cdot u_1 + N_2 \cdot u_2 + N_3 \cdot u_3 + N_4 \cdot u_4$$

$$v = N_1 \cdot v_1 + N_2 \cdot v_2 + N_3 \cdot v_3 + N_4 \cdot v_4$$

$$w = N_{11} \cdot w_1 + N_{12} \cdot \varphi_1 + N_{13} \cdot \Theta_1 + N_{21} w_2 + N_{22} \cdot \varphi_2 + N_{23} \Theta_2 + $$
$$+ N_{31} w_3 + N_{32} \varphi_3 + N_{33} \Theta_3 + N_{41} w_4 + N_{42} \varphi_4 + N_{43} \Theta_4$$

$$\underset{1\times 4}{\lfloor N \rfloor} = \lfloor N_1, N_2, N_3, N_4 \rfloor$$ (polynomials of ξ and η)

$$\underset{1\times 12}{\lfloor N_w \rfloor} = \lfloor N_{11}, N_{12}, N_{13}, N_{21}, N_{22}, N_{23}, N_{31}, N_{32}, N_{33}, N_{41}, N_{42}, N_{43} \rfloor$$

(Hermite polynomials)

Nodal approximation and shape functions

$$u = \underset{1\times 4}{\lfloor N \rfloor} \cdot \{q_u\}_e$$

$$v = \underset{1\times 4}{\lfloor N \rfloor} \{q_v\}_e$$

$$w = \underset{1\times 12}{\lfloor N_w \rfloor} \{q_w\}_e$$

$$\underset{3\times 1}{\{u\}} = \begin{Bmatrix} u \\ v \\ w \end{Bmatrix} = \underset{3\times 20}{\begin{bmatrix} \underset{1\times 4}{\lfloor N \rfloor} & \underset{1\times 4}{\lfloor 0 \rfloor} & \underset{1\times 12}{\lfloor 0 \rfloor} \\ \underset{1\times 4}{\lfloor 0 \rfloor} & \underset{1\times 4}{\lfloor N \rfloor} & \underset{1\times 12}{\lfloor 0 \rfloor} \\ \underset{1\times 4}{\lfloor 0 \rfloor} & \underset{1\times 4}{\lfloor 0 \rfloor} & \underset{1\times 12}{\lfloor N_w \rfloor} \end{bmatrix}} \cdot \underset{20\times 1}{\{q\}_e} = \underset{3\times 20}{[N]} \cdot \underset{20\times 1}{\{q\}_e}$$

MEMBRANE STRAIN:

$$\varepsilon_x^{MID} = \frac{\partial u}{\partial x} + k_1 \cdot w = \frac{\partial \underset{1\times 4}{\lfloor N \rfloor}}{\partial x} \cdot \{q_u\}_e + k_1 \cdot \underset{1\times 12}{\lfloor N_w \rfloor} \cdot \underset{12\times 1}{\{q_w\}_e}$$

$$\varepsilon_y^{MID} = \frac{\partial v}{\partial y} + k_2 \cdot w = \frac{\partial \underset{1\times 4}{\lfloor N \rfloor}}{\partial y} \{q_v\}_e + k_2 \cdot \underset{1\times 12}{\lfloor N_w \rfloor} \cdot \underset{12\times 1}{\{q_w\}_e}$$

$$\gamma_{xy}^{MID} = \frac{\partial u}{\partial v} + \frac{\partial v}{\partial x} + k_{12} \cdot w = \frac{\partial \underset{1\times 4}{\lfloor N \rfloor}}{\partial y} \cdot \{q_u\}_e + \frac{\partial \underset{1\times 4}{\lfloor N \rfloor}}{\partial x} \{q_v\}_e +$$

$$+ k_{12} \underset{1\times 12}{\lfloor N_w \rfloor} \cdot \underset{12\times 1}{\{q_w\}_e}$$

BENDING STRAIN (FUNCTION OF CURVATURES):

$$\kappa_x = -\frac{\partial^2 w}{\partial x^2} = -\frac{\partial^2 \underset{1\times 12}{\lfloor N_w \rfloor}}{\partial x^2} \cdot \underset{12\times 1}{\{q_w\}_e}$$

$$\kappa_y = -\frac{\partial^2 w}{\partial y^2} = -\frac{\partial^2 \underset{1\times 12}{\lfloor N_w \rfloor}}{\partial y^2} \cdot \{q_w\}_e$$

$$\kappa_{xy} = -2\frac{\partial^2 w}{\partial x \partial y} = -\frac{\partial^2 \underset{1\times 12}{\lfloor N_w \rfloor}}{\partial x \, \partial y} \{q_w\}_e$$

STRAIN- DISPLACEMENT MATRIX

$$
\underset{6\times1}{\left\{ \begin{array}{c} \underset{3\times1}{\{\varepsilon\}^{MID}} \\ \underset{3\times1}{\{\chi\}} \end{array} \right\}}
=
\underset{6\times20}{\left[\begin{array}{c|c|c}
\frac{\partial}{\partial x}\lfloor N \rfloor_{1\times4} & \lfloor 0 \rfloor_{1\times4} & K_1 \lfloor N_w \rfloor_{1\times12} \\
\hline
\lfloor 0 \rfloor_{1\times4} & \frac{\partial}{\partial y}\lfloor N \rfloor_{1\times4} & K_2 \lfloor N_w \rfloor_{1\times12} \\
\hline
\frac{\partial}{\partial y}\lfloor N \rfloor_{1\times4} & \frac{\partial}{\partial x}\lfloor N \rfloor_{1\times4} & K_{12} \lfloor N_w \rfloor_{1\times12} \\
\hline
 & & \frac{\partial}{\partial x}\left(\frac{\partial}{\partial x}\lfloor N_w \rfloor_{1\times12}\right) \\
\left[0\right]_{3\times4} & \left[0\right]_{3\times4} & \frac{\partial}{\partial y}\left(\frac{\partial}{\partial y}\lfloor N_w \rfloor_{1\times12}\right) \\
 & & -2\frac{\partial}{\partial y}\left(\frac{\partial}{\partial x}\lfloor N_w \rfloor_{1\times12}\right)
\end{array} \right]}
\cdot
\underset{20\times1}{\left\{ \begin{array}{c} \{q_u\}_{4\times1} \\ \{q_v\}_{4\times1} \\ \{q_w\}_{12\times1} \end{array} \right\}}
$$

STRAIN- DISPLACEMENT MATRIX

$$\underset{8\times 1}{\{q_{uv}\}_e} = \begin{Bmatrix} \underset{4\times 1}{\{q_u\}_e} \\ \underset{4\times 1}{\{q_v\}_e} \end{Bmatrix}_e$$

$$\underset{6\times 1}{\begin{Bmatrix} \underset{3\times 1}{\{\varepsilon\}^{MID}} \\ \underset{3\times 1}{\{\mathcal{H}\}} \end{Bmatrix}} = \left[\begin{array}{c|c} \underset{3\times 8}{[B_M]} & \underset{3\times 12}{[B_S]} \\ \hline \underset{3\times 8}{[0]} & \underset{3\times 12}{[B_B]} \end{array}\right] \cdot \begin{Bmatrix} \underset{8\times 1}{\{q_{uv}\}_e} \\ \underset{12\times 1}{\{q_w\}_e} \end{Bmatrix}_e$$

strain – displacement matrix

STRAIN- DISPLACEMENT MATRIX

$$\underset{3\times 8}{[B_M]} = \begin{bmatrix} \frac{\partial}{\partial x}\lfloor N \rfloor_{1\times 4} & \lfloor 0 \rfloor_{1\times 4} \\ \lfloor 0 \rfloor_{1\times 4} & \frac{\partial}{\partial y}\lfloor N \rfloor_{1\times 4} \\ \frac{\partial}{\partial y}\lfloor N \rfloor_{1\times 4} & \frac{\partial}{\partial x}\lfloor N \rfloor_{1\times 4} \end{bmatrix}$$

(middle layer)

$$\underset{3\times 12}{[B_S]} = \begin{bmatrix} k_1 \lfloor N_w \rfloor_{1\times 12} \\ k_2 \lfloor N_w \rfloor_{1\times 12} \\ k_{12} \lfloor N_w \rfloor_{1\times 12} \end{bmatrix}$$

(shell curvatures)

$$\underset{3\times 12}{[B_B]} = \begin{bmatrix} -\frac{\partial}{\partial x}\left(\frac{\partial}{\partial x}\lfloor N_w \rfloor_{1\times 12}\right) \\ -\frac{\partial}{\partial y}\left(\frac{\partial}{\partial y}\lfloor N_w \rfloor_{1\times 12}\right) \\ -2\frac{\partial}{\partial y}\left(\frac{\partial}{\partial x}\lfloor N_w \rfloor_{1\times 12}\right) \end{bmatrix}$$

(bending)

ELASTIC STRAIN ENERGY

$$\mathcal{U}_e = \mathcal{U}_e\left(\underset{3\times 1}{\{\varepsilon\}}^{MID}\right) + \mathcal{U}_e\left(\underset{3\times 1}{\{\mathcal{H}\}}\right)$$

$$\mathcal{U}_e\left(\underset{3\times 1}{\{\varepsilon\}}^{MID}\right) = \int_{A_e} \frac{1}{2} \underset{1\times 3}{\lfloor \varepsilon \rfloor}^{MID} \cdot \underset{3\times 1}{\{n\}} \, dA_e =$$

$$= \frac{1}{2}\int_{A_e} \underset{1\times 20}{\lfloor q \rfloor_e} \cdot \begin{bmatrix} \underset{8\times 3}{[B_M]^T} \\ \underset{12\times 3}{[B_S]^T} \end{bmatrix} \cdot \underset{3\times 3}{[D_n]} \cdot \underset{3\times 1}{\{\varepsilon^{MID}\}} \, dA_e =$$

ELASTIC STRAIN ENERGY

$\Rightarrow$

$$U_e\left(\underset{3\times1}{\{\varepsilon\}}^{MID}\right) = \frac{1}{2}\underset{1\times20}{\lfloor q \rfloor_e} \cdot \int_{A_e} \begin{bmatrix} \underset{8\times3}{[B_M]^T} \\ \underset{12\times3}{[B_S]^T} \end{bmatrix} \underset{3\times3}{[D_n]} \cdot \left[\underset{3\times8}{[B_M]} \;\vdots\; \underset{3\times12}{[B_S]} \right] dA_e \cdot \underset{20\times1}{\{q\}_e} =$$

$$= \frac{1}{2}\lfloor q \rfloor_e \int_{A_e} \begin{bmatrix} \underset{8\times3}{[B_M]^T} \\ \underset{12\times3}{[B_S]^T} \end{bmatrix} \cdot \left[\underset{3\times3}{[D_n]} \cdot \underset{3\times8}{[B_M]} \;\vdots\; \underset{3\times3}{[D_n]} \cdot \underset{3\times12}{[B_S]} \right] dA_e \; \underset{20\times1}{\{q\}_e} =$$

$$= \frac{1}{2}\lfloor q \rfloor_e \int_{A_e} \left[\begin{array}{c|c} \underset{8\times3}{[B_M]^T} \underset{3\times3}{[D_n]} \underset{3\times8}{[B_M]} & \underset{8\times3}{[B_M]^T} \cdot \underset{3\times3}{[D_n]} \cdot \underset{3\times12}{[B_S]} \\ \hline \underset{12\times3}{[B_S]^T} \underset{3\times3}{[D_n]} \underset{3\times8}{[B_M]} & \underset{12\times3}{[B_S]^T} \underset{3\times3}{[D_n]} \cdot \underset{3\times12}{[B_S]} \end{array} \right] dA_e \; \underset{20\times1}{\{q\}_e}$$

ELASTIC STRAIN ENERGY

$$U_e\left(\underset{3\times1}{\{\varkappa\}}\right) = \int_{A_e} \frac{1}{2} \cdot \underset{1\times3}{\lfloor \varkappa \rfloor} \cdot \underset{3\times1}{\{m\}} \, dA =$$

$$= \frac{1}{2} \int_{A_e} \underset{1\times20}{\lfloor q \rfloor_e} \cdot \begin{bmatrix} \underset{8\times3}{[0]} \\ \underset{12\times3}{[B_B]}^T \end{bmatrix} \cdot \underset{3\times3}{[D_m]} \cdot \underset{3\times1}{\{\varkappa\}} \, dA =$$

$$= \frac{1}{2} \underset{1\times20}{\lfloor q \rfloor_e} \int_{A_e} \begin{bmatrix} \underset{8\times3}{[0]} \\ \underset{12\times3}{[B_B]}^T \end{bmatrix} \underset{3\times3}{[D_m]} \cdot \left[\underset{3\times8}{[0]} \; \vdots \; \underset{3\times12}{[B_B]} \right] dA_e \cdot \underset{20\times1}{\{q\}_e} =$$

ELASTIC STRAIN ENERGY

$$\Rightarrow$$

$$U_e\left(\underset{3\times1}{\{\mathcal{H}\}}\right) = \frac{1}{2}\underset{1\times20}{\lfloor q \rfloor_e} \int_{A_e} \begin{bmatrix} \underset{8\times3}{[0]} \\ \underset{12\times3}{[B_B]^T} \end{bmatrix} \cdot \left[\underset{3\times3}{[D_m]}\underset{3\times8}{[0]} \;\vdots\; \underset{3\times3}{[D_m]} \cdot \underset{3\times12}{[B_B]} \right] dA_e \cdot \underset{20\times1}{\{q\}_e} =$$

$$= \frac{1}{2}\lfloor q \rfloor_e \int_{A_e} \left[\begin{array}{c|c} \underset{8\times8}{[0]} & \underset{8\times12}{[0]} \\ \hline \underset{12\times8}{[0]} & \underset{12\times3}{[B_B]^T}\underset{3\times3}{[D_m]}\underset{3\times12}{[B_B]} \end{array} \right] dA_e \{q\}_e \Rightarrow$$

ELASTIC STRAIN ENERGY

$$\Rightarrow U_e = \frac{1}{2} \underset{1\times 20}{\lfloor q \rfloor_e} \cdot \underset{20\times 20}{[k]_e} \cdot \underset{20\times 1}{\{q\}_e} \qquad \text{where:}$$

$$[k]_e = \int_{A_e} \left[\begin{array}{c|c} \underset{8\times 3}{[B_M]^T} \underset{3\times 3}{[D_n]} \underset{3\times 8}{[B_M]} & \underset{8\times 3}{[B_M]^T} \underset{3\times 3}{[D_n]} \underset{3\times 12}{[B_S]} \\ \hline \underset{12\times 3}{[B_S]^T} \underset{3\times 3}{[D_n]} \underset{3\times 8}{[B_M]} & \underset{12\times 3}{[B_S]^T} \underset{3\times 3}{[D_n]} \underset{3\times 12}{[B_S]} + \underset{12\times 3}{[B_B]^T} \underset{3\times 3}{[D_m]} \underset{3\times 12}{[B_B]} \end{array} \right] dA_e$$

Stiffness matrix of a shell finite element

POTENTIAL ENERGY OF LOADING

$$W_e = \underset{1\times 20}{\lfloor q \rfloor_e} \cdot \underset{20\times 1}{\{F\}_e}$$